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LIVŠIĆ THEOREM FOR DIFFEOMORPHISM COCYCLES

ARTUR AVILA, ALEJANDRO KOCSARD, AND XIAO-CHUAN LIU

ABSTRACT. We prove the so called Livšić theorem for cocycles taking values in the group of $C^{1+\beta}$ -diffeomorphisms of any closed manifold of arbitrary dimension. Since no localization hypothesis is assumed, this result is completely global in the space of cocycles and thus extends the previous result of the second author and Potrie [KP16] to higher dimensions.

1. INTRODUCTION

Given a dynamical system $f: M \rightarrow M$ and a map $A: M \rightarrow G$, where G denotes a topological group, a fundamental problem in the modern theory of dynamical systems consists in determining whether there exists a function $u: M \rightarrow G$ such that

$$A(x) = u(f(x)) (u(x))^{-1}, \quad \forall x \in M.$$

Such an equation is usually called a *cohomological equation* and when it does admit a solution, the cocycle A is said to be a *coboundary* for f .

A celebrated theorem of Livšić [Liv71, Liv72a, Liv72b] claims that when f is a hyperbolic homeomorphism, the group G is the additive real line \mathbb{R} and A is Hölder continuous, then there exists a continuous solution $u: M \rightarrow \mathbb{R}$ satisfying

$$(1) \quad A(x) = u \circ f - u,$$

if and only if

$$(2) \quad \sum_{i=0}^{n-1} A(f^i(p)) = 0, \quad \forall p \in \text{Fix}(f^n), \quad \forall n \geq 1.$$

Many generalizations have been studied. Some of them consider more general dynamical systems on the base. For example, in [KK96] and [Wil13] cohomological equations over partially hyperbolic diffeomorphisms are studied. Other generalizations deal with cocycles taking values in more general groups (see for instance [dlLW10, dlLW10]).

In this paper, we consider the second kind of generalization. When G is an arbitrary group, the above condition in Livšić theorem becomes

$$A(f^{n-1}(p)) \cdots A(p) = e_G,$$

for any periodic point p of period n , and where e_G denotes the identity element of the group G .

Livšić himself has already considered more general groups in [Liv71]. In fact, in that work he deals with cocycles taking values in a topological groups admitting a complete bi-invariant distance (e.g. Abelian or compact groups), and in this case, the very same techniques that were applied to the real case (i.e. when $G = \mathbb{R}$) still work.

Looking for generalizations of these results to cocycles taking values in groups not admitting bi-invariant distances, many authors (for instance [Liv72a, dlLW10, dlLW11]) have applied “distortion control techniques” which consist in endowing G

with a left-invariant distance and to estimate the distortion by right multiplication. However, in most of the cases this method alone only works under certain “localization hypothesis”, i.e. assuming that the cocycle is sufficiently close to the constant identity cocycle in certain topology, and therefore, does not produce global results.

The first global Livšic like theorem for topological groups not admitting a bi-invariant distance is due to Kalinin [Kal11], who proved Livšic theorem for linear cocycles substituting the localization hypothesis by some estimates of Lyapunov exponents.

In the infinite-dimensional case, that is when G is a topological group of infinite dimension, Nițică and Török [NT95, NT96] got the first local results (i.e. under some localization hypothesis) for groups of diffeomorphisms of closed manifolds. Some extensions of these results were obtained by de la Llave and Windsor in [dLLW10], but always under certain localization hypotheses. Navas and Ponce dealt in [NP13] with the case of cocycles taking values in the group of analytic germs at the origin.

The first global Livšic type theorem for groups of diffeomorphisms was recently obtained by Potrie and the second author of this paper in [KP16]. There they got a sharp characterization of coboundaries in terms of Lyapunov exponents, introducing some new geometric arguments and avoiding the usage of control distortion techniques. With this characterization of coboundaries, that holds for groups of diffeomorphisms of manifolds of any dimension, they proved a Livšic type theorem for cocycles with values in the group of circle C^1 -diffeomorphisms.

The main result of this paper is the so called Livšic theorem for Hölder cocycles taking values in the group of C^r -diffeomorphisms of a closed manifold, with $r > 1$, extending the previous result of [KP16] to higher dimensions.

1.1. Main results. All along this work, M will denote a compact metric space, $f: M \hookrightarrow M$ a hyperbolic homeomorphism, N a closed smooth manifold of dimension q and $\text{Diff}^r(N)$ the group of C^r -diffeomorphisms of N (see § 2 for details and definitions).

The main result of this work is the following

Theorem 1.1. *Let $A: M \rightarrow \text{Diff}^{1+\beta}(N)$ be a Hölder continuous cocycle, with $\beta > 0$. Then, A satisfies the periodic orbit condition, i.e. ,*

$$(3) \quad A(f^{n-1}(p)) \cdots A(f(p))A(p) = \text{id}_N, \quad \forall p \in \text{Fix}(f^n), \quad \forall n \geq 1,$$

if and only if there exists a Hölder continuous map $u: M \rightarrow \text{Diff}^{1+\beta}(N)$ such that

$$A(x) = u(f(x)) \circ u(x)^{-1}, \quad \forall x \in M.$$

As a consequence of Theorem 1.1 and some results due to de la Llave and Windsor [dLLW11], we get the Livšic theorem for groups of diffeomorphisms in higher regularity:

Corollary 1.2. *Let $A: M \rightarrow \text{Diff}^r(N)$ be a Hölder continuous cocycle, with $r > 1$. Then, A satisfies the periodic orbit condition (3) if and only if there exists a Hölder continuous map $u: M \rightarrow \text{Diff}^r(N)$ such that*

$$A(x) = u(f(x)) \circ u(x)^{-1}, \quad \forall x \in M.$$

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2. PRELIMINARIES AND NOTATIONS

2.1. $C^{1+\beta}$ -distance on groups of diffeomorphisms. All along this article N will denote a compact Riemannian manifold and we shall write d_N for its distance function induced by the Riemannian structure.

We shall consider the group of $C^{1+\beta}$ -diffeomorphisms of N , which is denoted by $\text{Diff}^{1+\beta}(N)$, endowed with the distance function $d_{C^{1+\beta}}$ which is defined as follows: first we define the d_{C^0} distance by

$$d_{C^0}(f, g) := \max \{d(f(x), g(x)) : x \in N\}, \quad \forall f, g \in \text{Diff}^{1+\beta}(N).$$

Then, since N is compact, there is finite atlas $\{\phi_i : U_i \subset N \rightarrow \mathbb{R}^q, i = 1, \dots, m\}$ and compact sets $K_i \subset U_i$ such that $\bigcup_{i=1}^m K_i = N$. So, given any pair of diffeomorphisms $f, g \in \text{Diff}^{1+\beta}(N)$ satisfying

$$(4) \quad f \circ g^{-1}(K_i), g \circ f^{-1}(K_i) \subset U_i, \quad \forall i \in \{1, \dots, m\},$$

we define the distance d_{C^1} by

$$d_{C^1}(f, g) := d_{C^0}(f, g) + \max_{1 \leq i \leq m} \left\{ \max_{x \in \phi_i(K_i)} \|D(\phi_i \circ f \circ g^{-1} \circ \phi_i^{-1}(x))\| \right\} \\ + \max_{1 \leq i \leq m} \left\{ \max_{x \in \phi_i(K_i)} \|D(\phi_i \circ g \circ f^{-1} \circ \phi_i^{-1}(x))\| \right\}.$$

Then, we define the distance $d_{C^{1+\beta}}$ by

$$d_{C^{1+\beta}}(f, g) := d_{C^1}(f, g) + \max_{1 \leq i \leq m} \left\{ \max_{x, y \in \phi_i(K_i), x \neq y} \frac{\|D(\phi_i \circ f \circ g^{-1} \circ \phi_i^{-1}(x)) - D(\phi_i \circ f \circ g^{-1} \circ \phi_i^{-1}(y))\|}{\|x - y\|^\beta} \right\} + \\ \max_{1 \leq i \leq m} \left\{ \max_{x, y \in \phi_i(K_i), x \neq y} \frac{\|D(\phi_i \circ g \circ f^{-1} \circ \phi_i^{-1}(x)) - D(\phi_i \circ g \circ f^{-1} \circ \phi_i^{-1}(y))\|}{\|x - y\|^\beta} \right\}.$$

Finally, we define $d_{C^1}(f, g) = d_{C^0}(f, g) + 1$ and $d_{C^{1+\beta}}(f, g) = d_{C^1}(f, g) + 1$, whenever condition (4) does not hold.

2.2. Hyperbolic homeomorphisms. Given a compact metric space (M, d) , a homeomorphism $f : M \rightarrow M$ and an arbitrary point $x \in M$, let us consider the *local stable and unstable sets* at x given by

$$W_\varepsilon^s(x) := \{y \in M : d(f^n(x), f^n(y)) \leq \varepsilon, \forall n \geq 0\}; \\ W_\varepsilon^u(x) := \{y \in M : d(f^{-n}(x), f^{-n}(y)) \leq \varepsilon, \forall n \geq 0\},$$

where ε is any positive real number.

Definition 2.1. The homeomorphism f is said to be *hyperbolic* whenever it is bi-Lipschitz (i.e. f and f^{-1} are Lipschitz), transitive and there exist constants $\varepsilon, \delta, K_0, \tau > 0$ and continuous real functions $\nu_s, \nu_u : M \rightarrow (0, \infty)$ such that the following properties hold:

- (i) $d(f(y'), f(y'')) \leq \nu_s(y)d(y', y'')$ for any $y', y'' \in W_\varepsilon^s(y)$ and any $y \in M$;
- (ii) $d(f^{-1}(y'), f^{-1}(y'')) \leq \nu_u(y)d(y', y'')$ for any $y', y'' \in W_\varepsilon^u(y)$ and any $y \in M$;
- (iii) $\nu_s^{(n)}(y) = \nu_s(f^{n-1}(y)) \cdots \nu_s(y) \leq K_0 e^{-\tau n}$ for any $y \in M$ and any $n \geq 1$;
- (iv) $\nu_u^{(n)}(y) = \nu_u(f^{-(n-1)}(y)) \cdots \nu_u(y) \leq K_0 e^{-\tau n}$ for any $y \in M$ and any $n \geq 1$;
- (v) if $d(y, y') < \delta$, then $W_\varepsilon^u(y) \cap W_\varepsilon^s(y')$ consists of exactly one point, denoted by $[y, y']$, which depends continuously on (y, y') ,

A fundamental property about hyperbolic homeomorphisms is given by the so called Anosov closing lemma (see for instance [KH96, Theorem 6.4.15]):

Lemma 2.2. *If $f: M \hookrightarrow M$ is a hyperbolic homeomorphism, then there exist constants $C, \delta > 0$ such that for any $x \in M$ and any $n \geq 1$ such that $d(x, f^n(x)) < \delta$, there exists a unique periodic point $p \in \text{Fix}(f^n)$ satisfying*

$$d(f^i(x), f^i(p)) \leq Cd(x, f^n(x))e^{-\tau \min(i, n-i)} \text{ for any } i = 0, \dots, n,$$

where τ is the constant given by Definition 2.1.

Moreover, if we define $y := [p, x]$, it holds

$$d(f^i(x), f^i(y)) \leq Cd(x, f^n(x))e^{-\tau i},$$

and

$$d(f^i(p), f^i(y)) < Cd(x, f^n(x))e^{-\tau(n-i)},$$

for any $i = 0, 1, \dots, n$.

2.3. Cocycles and coboundaries. Let G be a topological group and $f: M \hookrightarrow M$ be a hyperbolic homeomorphism. Every continuous map $A: M \rightarrow G$ shall be considered as a G -cocycle over f , i.e. we will use the following notation: given any $n \in \mathbb{Z}$, we write

$$A^n(x) := \begin{cases} e_G, & \text{if } n = 0; \\ A(f^{n-1}(x)) \cdots A(x), & \text{if } n \geq 1, \\ A(f^n(x))^{-1} \cdots A(f^{-1}(x))^{-1}, & \text{if } n < 0. \end{cases}$$

It is important to notice that any G -cocycle over $f: M \hookrightarrow M$ naturally induces a skew-product map $F = (f, A): M \times G \hookrightarrow M \times G$ given by

$$(5) \quad (f, A)(x, g) := (f(x), A(x)g), \quad \forall (x, g) \in M \times G.$$

Observe that in this case, it holds $(f, A)^n(x, g) = (f^n(x), A^n(x)g)$, for every $n \in \mathbb{Z}$, and any $(x, g) \in M \times G$.

Definition 2.3. A G -cocycle A is called a *coboundary* whenever there exists a continuous mapping $u: M \rightarrow G$ satisfying

$$(6) \quad A(x) = u(f(x)) \cdot u(x)^{-1}, \quad \forall x \in M.$$

An obvious, yet extremely important, family of obstructions for a G -cocycle to be a coboundary is given by periodic orbits of f . In fact, if A is a coboundary, it necessarily holds

$$(7) \quad A^n(p) = e_G, \quad \forall n \in \mathbb{N}, \quad \forall p \in \text{Fix}(f^n).$$

In this paper we will say that a G -cocycle A satisfies the *periodic orbit condition* (POC for short) whenever property (7) holds.

2.4. Fibered Lyapunov exponents. In this work we are mainly interested in the study of G -cocycles with $G < \text{Diff}^r(N)$, where N is a closed smooth manifold.

From now on and till the end of the paper we will suppose that N endowed with a Riemannian structure $\langle \cdot, \cdot \rangle$, and the norm induced by the Riemannian structure will be denoted by $\|\cdot\|$.

In this case, when G is a subgroup of the $\text{Diff}^r(N)$, we can slightly modify the skew-product given by (5) defining $F = (f, A): M \times N \hookrightarrow M \times N$ by

$$(8) \quad F(x, y) := (f(x), A(x)y), \quad \forall (x, y) \in M \times N.$$

Observe that the map F is C^r along the vertical fibers. So, we can consider the “partial derivative of F respect to the N coordinate”, defining the linear cocycle over F given by

$$(9) \quad \partial F_{(x,y)}^n v := D(A^n(x))(y)(v), \quad \forall x \in M, \quad \forall y \in N, \quad \forall v \in T_y N,$$

and any $n \in \mathbb{Z}$.

In this way we can apply Oseledets theorem to guarantee that, for every ergodic F -invariant probability measure μ , there exist $k \leq \dim N$, real numbers $\lambda'_1 > \lambda'_2 > \dots > \lambda'_k$ and a measurable F -invariant set $\Lambda \subset M \times N$ with $\mu(\Lambda) = 1$, such that for every $z = (x, y) \in \Lambda$ there exists a linear splitting $T_y N = E_z^1 \oplus E_z^2 \oplus \dots \oplus E_z^k$ which measurably depends on z , satisfies $\partial F_z(E_z^i) = E_{F(z)}^i$, for every $z \in \Lambda$ and every $1 \leq i \leq k$, and

$$(10) \quad \lambda'_i := \lim_{n \rightarrow \pm\infty} \frac{1}{n} \log \|\partial F_z^n(v)\|, \quad \forall z \in \Lambda, \quad \forall v \in E_z^i \setminus \{0\}, \quad \forall i \leq k.$$

2.5. Lyapunov exponents and coboundaries. As it was already shown in [KP16], when dealing with cocycle over hyperbolic homeomorphisms with values in groups of diffeomorphisms there is a very tight relation between Lyapunov exponents and coboundaries. In fact, the main result of [KP16] is the following:

Theorem 2.4 (Theorem 3.1 in [KP16]). *Let $f: M \hookrightarrow$ be a hyperbolic homeomorphism and $A: M \rightarrow \text{Diff}^1(N)$ be a Hölder continuous cocycle over f satisfying the POC given by (7).*

Then the following statements are equivalent:

- (1) *Every Lyapunov exponent given by (10) with respect to any F -invariant measure vanish, i.e. it holds*

$$\lim_{n \rightarrow \pm\infty} \frac{1}{n} \log \|\partial F_z^n\| = 0, \quad \forall z \in M \times N;$$

- (2) *The cocycle A is a coboundary, i.e. there exists a Hölder continuous map $u: M \rightarrow \text{Diff}^1(N)$ such that*

$$A(x) = u(f(x)) \circ u(x)^{-1}, \quad \forall x \in M.$$

Remark. We should note that, for our main statement, we have to consider $C^{1+\beta}$ topology instead of C^1 topology as in the above statement. The reason is that the proof involves techniques from Pesin theory.

So, in order to show Theorem 1.1 it is enough to prove the following

Theorem 2.5. *If $f: M \hookrightarrow$ is a hyperbolic homeomorphism and $A: M \rightarrow \text{Diff}^{1+\beta}(N)$ is a Hölder cocycle satisfying POC, then*

$$\lim_{n \rightarrow \pm\infty} \frac{1}{n} \log \|\partial F_z^n\| = 0, \quad \forall z \in M \times N.$$

It is worth mentioning that in [KP16] it is considered a slightly more general setting, dealing with *a priori* arbitrary fiber bundles. There it is shown that the fiber bundle is *a posteriori* trivial whenever the corresponding fiber bundle map turns to be a coboundary (see § 2.6 in [KP16] for more details).

Since all our techniques are either local or semi-local, we could have worked in the fiber bundle category, getting the very same results. However, in order to avoid cumbersome unnecessary technicalities we have decided to present all our work in product spaces, i.e. *a priori* trivial fiber bundles.

3. SOME LINEAR ALGEBRAIC LEMMAS

Let us consider \mathbb{R}^d endowed with its usual Euclidean structure. Let us denote the closed ball of radius r and centered at the origin by $B_r := \{x \in \mathbb{R}^d : \|x\| \leq r\}$ and the sphere of radius r by $S_r := \{x \in \mathbb{R}^d : \|x\| = r\}$.

Given an ellipsoid $E \subset \mathbb{R}^d$, we shall write $r_1(E) \geq r_2(E) \geq \dots \geq r_d(E)$ for the ordered lengths of its semi-axes.

Our first result of this section is the following elementary

Lemma 3.1. *If $E, F \subset \mathbb{R}^d$ are two ellipsoids centered at the origin and such that E is contained in the interior of F , then it holds*

$$r_i(E) \leq r_i(F), \quad \forall i \in \{1, \dots, d\}.$$

Proof. Let $\{e_1, e_2, \dots, e_d\}$ and $\{f_1, f_2, \dots, f_d\}$ be two orthonormal bases of \mathbb{R}^d such that the directions of e_i and f_i coincide with the direction of the i^{th} -axis of E and F , respectively.

First notice that, since E is contained in the interior of F , it clearly holds $r_1(E) \leq r_1(F)$. Reasoning by contradiction, let us suppose there exists $j \in \{2, \dots, d\}$ such that $r_i(E) \leq r_i(F)$, for every $i \leq j-1$ and $r_j(E) > r_j(F)$. Let $V_{E,j} := \text{span}\{e_1, \dots, e_j\}$ and $V_F^j := \text{span}\{f_j, \dots, f_d\}$. Notice that $\|v\| \geq r_j(E)$, for all $v \in E \cap V_{E,j}$; and $\|v\| \leq r_j(F)$, for every $v \in F \cap V_F^j$. However, $\dim V_{E,j} + \dim V_F^j = j + d - j + 1 = d + 1$. So, there exists some $v \in E \cap V_{E,j} \cap V_F^j$ and consequently, $\|v\| \geq r_j(E) > r_j(F)$, contradicting the fact that E is contained in the interior of F . \square

For any $A \in \text{GL}(d, \mathbb{R})$, the image of the unit sphere $A(\mathbb{S}_1)$ is an ellipsoid centered at the origin, and the *singular values* of A are defined by

$$\sigma_i(A) := r_i(A(\mathbb{S}_1)), \quad \text{for } i \in \{1, \dots, d\}.$$

It is well-known that $\sigma_i(A)^2$ is an eigenvalue for AA^t , where A^t denotes the transpose of A .

Our first perturbative result is the following:

Lemma 3.2. *Let $(A_n)_{n \geq 1}$ be a sequence of matrices in $\text{GL}(d, \mathbb{R})$ and suppose there exist real numbers $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_d$ and $\delta > 0$ such that*

$$\left| \frac{1}{n} \log(\sigma_i(A_n)) - \lambda_i \right| \leq \frac{\delta}{2}, \quad \forall n \geq 1, \quad \forall i \in \{1, \dots, d\}.$$

Then, given any compact set $K \subset \text{GL}(d, \mathbb{R})$, there exists $N = N(K) > 0$ such that for every pair of sequences $(C_n)_{n \geq 1}, (D_n)_{n \geq 1} \subset K$, it holds

$$\left| \frac{1}{n} \log(\sigma_i(C_n A_n D_n)) - \lambda_i \right| \leq \delta, \quad \forall n \geq N,$$

and every $i \in \{1, \dots, d\}$.

Proof. Since K is compact, there exists $\ell > 1$ such that

$$\max\{\|C\|, \|C^{-1}\|\} \leq \ell, \quad \forall C \in K.$$

So, if E is an arbitrary ellipsoid centered at the origin, then we have $\frac{1}{\ell}E \subset C(E) \subset \ell E$, and by Lemma 3.1 it holds

$$\frac{1}{\ell} r_i(E) \leq r_i(C(E)) \leq \ell r_i(E), \quad \forall C \in K,$$

and every i .

Consequently, for any $n \geq 1$ it clearly holds

$$\frac{1}{\ell^2} \sigma_i(A_n) \leq \sigma_i(C_n A_n D_n) \leq \ell^2 \sigma_i(A_n), \quad \forall n \geq 1,$$

and every i .

So,

$$\begin{aligned} -\frac{2 \log \ell}{n} + \frac{1}{n} \log \sigma_i(A_n) - \lambda_i &\leq \frac{1}{n} \log(\sigma_i(C_n A_n D_n)) - \lambda_i \\ &\leq \frac{2 \log \ell}{n} + \sigma_i(A_n) - \lambda_i, \end{aligned}$$

for every n and any i , and taking $N := \frac{4 \log \ell}{\delta}$ the lemma is proved. \square

Lemma 3.3. *Let $\mathbb{R}^d = \bigoplus_{i=1}^k E^i$ be an orthogonal linear decomposition and write $d_i := \dim E^i$. Consider real numbers $\lambda'_1 > \lambda'_2 > \dots > \lambda'_k$, define*

$$\kappa := \frac{1}{2} \min_{1 \leq i \leq k-1} \{\lambda'_i - \lambda'_{i+1}\},$$

and let δ be any number satisfying $0 < \delta < \kappa/2$.

Let $(A_n)_{n \geq 1} \subset \text{GL}(d, \mathbb{R})$ be a sequence of matrices such that

$$(11) \quad A_n(E^i) = E^i,$$

and

$$(12) \quad \|A_n|_{E^i}\|, \|A_n^{-1}|_{E^i}\|^{-1} \in \left(e^{\lambda'_i - \frac{\delta}{4}}, e^{\lambda'_i + \frac{\delta}{4}}\right), \quad \forall n \geq 1,$$

and every $i \in \{1, \dots, k\}$.

Then, there exist $\alpha_1 = \alpha_1(\delta) > 0$ so that for every $(C_n)_{n \geq 1} \subset \text{GL}(d, \mathbb{R})$ satisfying

$$\|A_n - C_n\| < \alpha_1, \quad \forall n \geq 1,$$

it holds

$$(13) \quad \left| \frac{1}{n} \log \sigma_i(A^{(n)}) - \frac{1}{n} \log \sigma_i(C^{(n)}) \right| < \delta, \quad \forall n \geq 1, \quad \forall i \in \{1, \dots, k\},$$

where $A^{(n)} := \prod_{j=1}^n A_j$ and $C^{(n)} := \prod_{j=1}^n C_j$.

Proof. Given any $v \in \mathbb{R}^d$, we write $v = \sum_{i=1}^k v^i$ for its unique decomposition with $v^i \in E^i$, for each i .

For each $j \in \{1, \dots, k-1\}$ and $\gamma > 0$, let us consider the cones

$$K_\gamma^j := \left\{ v \in \mathbb{R}^d : \left\| \sum_{i \geq j+1} v^i \right\| < \gamma \left\| \sum_{i \leq j} v^i \right\| \right\},$$

and

$$K_{j,\gamma} := \left\{ v \in \mathbb{R}^d : \left\| \sum_{i \leq j} v^i \right\| < \gamma \left\| \sum_{i \geq j+1} v^i \right\| \right\}.$$

For each $j \in \{1, \dots, k-1\}$ and any n , since we are assuming the linear decomposition $\mathbb{R}^d = \bigoplus_i E^i$ is orthogonal, it holds

$$\begin{aligned} \left\| A_n \sum_{i \leq j} v^i \right\|^2 &= \sum_{i \leq j} \|A_n v^i\|^2 \geq \left(e^{\lambda'_j - \delta/2} \right)^2 \left\| \sum_{i \leq j} v^i \right\|^2 \\ &> (e^{\lambda'_{j+1} + \kappa})^2 \left\| \frac{1}{\gamma} \sum_{i \geq j+1} v^i \right\|^2 > \left(\frac{e^{\kappa - \delta/2}}{\gamma} \right)^2 \left\| A_n \sum_{i \geq j+1} v^i \right\|^2, \end{aligned}$$

for every $v \in K_\gamma^j$. Analogously, for every $v \in K_{j,\gamma}$ we get

$$\begin{aligned} \left\| A_n^{-1} \sum_{i \leq j} v^i \right\|^2 &= \sum_{i \leq j} \|A_n^{-1} v^i\|^2 \leq \left(e^{\lambda'_j - \delta/3} \right)^{-2} \left\| \sum_{i \leq j} v^i \right\|^2 \\ &< (e^{\lambda'_{j+1} + \kappa + \delta/12})^{-2} \left\| \gamma \sum_{i \geq j+1} v^i \right\|^2 < \left(\frac{\gamma}{e^{\kappa - \delta/4}} \right)^2 \left\| A_n^{-1} \sum_{i \geq j+1} v^i \right\|^2. \end{aligned}$$

This implies that

$$(14) \quad A_n(K_\gamma^j) \subset K_{\gamma e^{-\kappa + \delta/4}}^j, \quad \text{and} \quad A_n^{-1}(K_{j,\gamma}) \subset K_{j,\gamma e^{-\kappa + \delta/4}},$$

and for every $\gamma > 0$, each $j \in \{1, \dots, d-1\}$ and any $n \geq 1$.

Then let us fix $\gamma > 0$ sufficiently small such that

$$(15) \quad \|A_n v\| \leq e^{\lambda'_{j+1} + \delta/3} \|v\|, \quad \text{and} \quad \|A_n^{-1} u\| \geq e^{\lambda'_j - \delta/3} \|u\|$$

for every $j < k$, any $v \in K_{j,\gamma}$, all $u \in K_\gamma^j$ and every $n \geq 1$.

After fixing γ as above, and taking into account (14) and (15), we choose $\alpha_1 > 0$ small enough such that for any sequence of matrices $(C_n)_{n \geq 1} \subset \text{GL}(d, \mathbb{R})$ satisfying $\|A_n - C_n\| < \alpha_1$ for every n , it holds

$$(16) \quad \|C_n^{-1}\|^{-1} \geq e^{\lambda'_k - \delta/2},$$

$$(17) \quad C_n(K_\gamma^j) \subset K_{\gamma e^{-\kappa + \delta/2}}^j, \quad \text{and} \quad C_n^{-1}(K_{j,\gamma}) \subset K_{j,\gamma e^{-\kappa + \delta/2}}.$$

and

$$(18) \quad \|C_n v\| \leq e^{\lambda'_{j+1} + \delta/2} \|v\|, \quad \text{and} \quad \|C_n^{-1} u\| \geq e^{\lambda'_j - \delta/2} \|u\|,$$

for every $j \in \{1, \dots, d-1\}$, any $n \geq 1$, every $v \in K_{j,\gamma}$ and all $u \in K_\gamma^j$.

Then, for any such sequence $(C_n)_{n \geq 1}$, let us define

$$(19) \quad H_k := \bigcap_{n \geq 1} (C^{(n)})^{-1}(\overline{K_{k-1,\gamma}}).$$

By classical arguments one can easily show that H_k is a linear subspace satisfying $\dim H_k = \dim E^k$. Putting together (16), (18) and (19) one can easily verify that

$$e^{n(\lambda_k - \delta/2)} \|v\| \leq \|C^{(n)} v\| \leq e^{n(\lambda'_k + \delta/2)} \|v\|, \quad \forall v \in H^k.$$

On the other hand, for every $v \in \mathbb{R}^d \setminus H^k$, there exists $N \in \mathbb{N}$ such that $C^{(n)}(v) \in K_\gamma^{k-1}$, and thus one gets

$$\liminf_{n \rightarrow +\infty} \frac{1}{n} \log \|C^{(n)}(v)\| \geq \lambda'_{k-1} - \delta > \lambda'_k + \delta.$$

Then, we define

$$H_{k-1} := \bigcap_{n \geq 1} (C^{(n)})^{-1}(\overline{K_{k-2,\gamma}}),$$

and it can be shown that $H_{k-1} \supset H_k$ is a linear subspace satisfying $\dim H_{k-1} = \dim E^k + \dim E^{k-1}$, and by a similar argument to that used above, we get

$$e^{n(\lambda'_{k-1} - \delta/2)} \|v\| \leq \|C^{(n)}(v)\| \leq e^{n(\lambda'_{k-1} + \delta/2)} \|v\|, \quad \forall v \in H_{k-1} \setminus H_k.$$

Inductively one defines the flag $\mathbb{R}^d = H_1 \supset H_2 \supset \dots \supset H_k$ such that for any $v \in H_j \setminus H_{j+1}$, it holds

$$(20) \quad \lambda'_j - \frac{\delta}{2} \leq \liminf_{n \rightarrow +\infty} \frac{1}{n} \log \|C^{(n)}(v)\| \leq \limsup_{n \rightarrow +\infty} \frac{1}{n} \log \|C^{(n)}(v)\| \leq \lambda'_j + \frac{\delta}{2}.$$

Finally, by (11) and (12) we know that for each $i \in \{1, \dots, d\}$, there exists $j \in \{1, \dots, k\}$ such that

$$\left| \frac{1}{n} \sigma_i(A^{(n)}) - \lambda'_j \right| \leq \frac{\delta}{2},$$

and by (20) we also know that

$$\left| \frac{1}{n} \sigma_i(C^{(n)}) - \lambda'_j \right| \leq \frac{\delta}{2}.$$

Then, (13) is consequence of last two estimates and the lemma is proved. \square

4. FIBER-WISE PESIN THEORY AND FAKE INVARIANT SETS

The purpose of this section consists introducing the so called *fake invariant sets*, which are something like a “finite-time stable and unstable sets” and are mainly inspired by fake invariant foliations of Burns and Wilkinson [BW10] and pseudo-hyperbolic families of Hirsch, Pugh and Shub [HPS77]. We start stating a fiber-wise version of some classical notions and results due to Pesin, and we shall just sketch their proof.

Let $f: M \hookrightarrow M$ be a hyperbolic homeomorphism on the compact metric space (M, d_M) , $A: M \rightarrow \text{Diff}^{1+\beta}(N)$ a C^β -cocycle, for some $\beta > 0$, and $F: M \times N \hookrightarrow M \times N$ be the induced skew-product given by (8). Let d_N be the distance function on N induced by its Riemannian structure. We shall consider the space $M \times N$ endowed with the distance d given by

$$d(z, z') := \sqrt{d_M(x, x')^2 + d_N(y, y')^2}, \quad \forall z = (x, y), \quad z' = (x', y') \in M \times N.$$

Notice that, since A is a C^β -Hölder map, if $d_{C^{1+\beta}}$ denotes a distance on $\text{Diff}^{1+\beta}(N)$ inducing the uniform $C^{1+\beta}$ -topology, then there exists a real constant $K > 0$ such that

$$(21) \quad d_{C^{1+\beta}}(A(x), A(x')) \leq K d_M(x, x')^\beta, \quad \forall x, x' \in M,$$

where we use the distance $d_{C^{1+\beta}}$ defined in Section 2.1.

4.1. Fiber-wise Pesin charts. Consider the linear cocycle $\partial F: M \times TN \hookrightarrow M \times TN$ given by (9), which is in fact a fiber bundle map over F . Let us fix an arbitrary ergodic F -invariant probability measure μ .

Then, there exists a finite collection of open subsets $\Delta_1, \Delta_2, \dots, \Delta_{k_0}$ of N such that

$$\mu\left(M \times \left(N \setminus \bigcup_{i=1}^{k_0} \Delta_i\right)\right) = 0,$$

and the tangent bundle $TN|_{\Delta_i}$ is trivial, i.e. is diffeomorphic to $\Delta_i \times \mathbb{R}^q$, for every $i \in \{1, \dots, k_0\}$, where q denotes the dimension of N . Via a smooth trivialization of the tangent bundle TN over each Δ_i , we can identify, up to μ -measure zero, the derivative bundle map $\partial F: M \times TN \hookrightarrow M \times TN$ with a linear cocycle over $F: M \times N \hookrightarrow M \times N$ (for details see [KH96, Supplement, §2.b]).

In other words, from now on we shall simply assume that the bundle map ∂F is represented by a linear cocycle

$$(22) \quad B(z) := DA(x)(y) \in \text{GL}(q, \mathbb{R}), \quad \forall z = (x, y) \in M \times N,$$

which is well-defined μ -almost everywhere.

Now, let $r > 0$ be the injectivity radius of the Riemannian manifold N . For any linear cocycle $L: M \times N \rightarrow \text{GL}(q, \mathbb{R})$, let us define the injective map $\Phi_z: B(y, r) \rightarrow \mathbb{R}^q$ by

$$(23) \quad \Phi_z(y') := L(z) \circ \exp_y^{-1}(y'), \quad \forall y' \in B(y, r),$$

where $B(y, r) \subset \{x\} \times N$ is the ball of radius r and center y with respect to the distance d_N .

For $r' > 0$ sufficiently small (and smaller than r in the previous paragraph), from now on, $z = (x, y)$ and $z' = (x', y')$ will denote points of $M \times N$, where $x' \in B_M(x, r')$. Now, we are interested in the cocycle $L = B$, the derivative cocycle (22), Φ_z will be the corresponding function defined in (23), and hence, we can define the map $h_{z, z'}: B(0, r') \subset \mathbb{R}^q \rightarrow \mathbb{R}^q$ by

$$(24) \quad h_{z, z'}(v) := \Phi_{F(z)} \circ \text{pr}_2 \circ F(x', \Phi_z^{-1}(v)), \quad \forall v \in B(0, r').$$

Let $\lambda'_1 > \dots > \lambda'_k$ be the Lyapunov exponents of the linear cocycle B with respect to μ , where each exponent λ'_j has multiplicity n_j , $j = 1, \dots, k$. The following result is a slight generalization of the classical construction of Pesin charts:

Theorem 4.1. *For every $\eta > 0$, there exist a measurable set $\Lambda_\eta \subset M \times N$ with full μ -measure and a tempered map $C_\eta: M \times N \rightarrow \text{GL}(q, \mathbb{R})$, satisfying the following properties:*

- (1) *For any $z \in \Lambda_\eta$, the matrix $B_\eta(z) = C_\eta(F(z))B(z)C_\eta^{-1}(z)$ has the Lyapunov block form*

$$\text{diag}(B_{\eta,1}(z), \dots, B_{\eta,k}(z)),$$

where each $B_{\eta,j}(z)$ is an $n_j \times n_j$ square matrix, n_j denotes the multiplicity of the corresponding exponent λ'_j and the norm $\|B_{\eta,j}(z)\|$ and co-norm $\|B_{\eta,j}(z)^{-1}\|^{-1}$ lie in the interval $[e^{\lambda'_j - \eta}, e^{\lambda'_j + \eta}]$.

- (2) *There exists a measurable function $r_1: \Lambda_\eta \rightarrow (0, r']$, with*

$$(25) \quad e^{-\eta/\beta^2} < \frac{r_1(z)}{r_1(F(z))} < e^{\eta/\beta^2}, \quad \forall z \in \Lambda_\eta,$$

such that, for any point $z = (x, y) \in \Lambda_\eta$ and any $x' \in B_M(x, r_1(z))$, the map $h_{z',z}$ given by (24) where $z' = (x', y)$, satisfies

$$(26) \quad d_{C^1}(h_{z,z'}, h_{z,z}) \leq \eta.$$

- (3) *There exist measurable functions $K: \Lambda_\eta \rightarrow (0, \infty)$ and $r_2: \Lambda_\eta \rightarrow (0, r']$ with*

$$e^{-\eta} < \frac{K(z)}{K(F(z))} < e^\eta,$$

$$e^{-\eta/\beta} < \frac{r_2(z)}{r_2(F(z))} < e^{\eta/\beta},$$

such that, for every $z \in \Lambda_\eta$, and for any $y', y'' \in \Phi_z^{-1}(B(0, r_2(z)))$,

$$(27) \quad \frac{1}{2}d_N(y', y'') \leq \|\Phi_z(y') - \Phi_z(y'')\| \leq K(z)d_N(y', y'').$$

Proof. This result can be easily proved following *mutatis mutandis* the proof of [KH96, Theorem S.3.1]. The only significant difference is that to control our regular neighborhood we need two different radii, namely r_1 and r_2 , for the size of the balls on the base space and on the fibers. \square

Then we recall some classical definitions of Pesin theory. Continuing with the notations of Theorem 4.1, every point $z = (x, y)$ of Λ_η is called a *regular point*, and each set $B_M(x, r_1(z)) \times \Phi_z^{-1}(B(0, r_2(z)))$ is called a *regular neighborhood* of z .

On the other hand, for each $\eta > 0$ sufficiently small and any $\varepsilon > 0$, by Luzin's theorem, there exist $\ell > 0$ and a compact subset $\Lambda_{\eta,\ell} \subset \Lambda$ such that $\mu(\Lambda_{\eta,\ell}) > 1 - \varepsilon$, the functions $r_1|_{\Lambda_{\eta,\ell}}$, $r_2|_{\Lambda_{\eta,\ell}}$, $K|_{\Lambda_{\eta,\ell}}$ and $C_\eta|_{\Lambda_{\eta,\ell}}$ are continuous, and $|r_1(z)^{-1}| \leq \ell$ and $|K(z)| \leq \ell$, for any $z \in \Lambda_{\eta,\ell}$. The compact set $\Lambda_{\eta,\ell}$ is called the *Pesin uniformity block* of tolerance η and bound ℓ .

4.2. Fake Invariant Sets. For the sake of simplicity of the notation, given an arbitrary point $z_0 = (x_0, y_0) \in M \times N$, we write $z_n = (x_n, y_n) = F^n(z_0)$, for every $n \in \mathbb{Z}$. Given any positive number r we write

$$U = U(z_n, r) := B_M(x_n, r) \times B_N(y_n, r),$$

Continuing with the notation we introduced in §4.1, let μ denote an ergodic F -invariant measure, η be a sufficiently small positive number and ℓ a sufficiently large

positive number, such that the Pesin uniformity block $\Lambda_{\eta,\ell} \subset \Lambda_\eta$ has positive μ -measure. Hence by Poincaré recurrence theorem, for μ -almost every point $z_0 \in \Lambda_{\eta,\ell}$, there exists a positive integer number N_0 such that $z_{N_0} \in \Lambda_{\eta,\ell}$.

Now we can state the main result of this section:

Proposition 4.2 (Existence of fake invariant sets). *There exist positive real constants $r^{(0)}, \kappa, C, \tilde{C}$ such that, given any pair of points $z_0, z_{N_0} \in \Lambda_{\eta,\ell}$ (for some $N_0 > 1$), there exists a partition $\hat{\mathcal{U}}^{n,s}$ of $U(z_n, r^{(n)})$, for each $n \in \{0, 1, \dots, N_0\}$, where*

$$(28) \quad r^{(n)} := r^{(0)} e^{-\frac{\eta}{\beta^2} \min\{n, N_0 - n\}},$$

satisfying the following properties:

- (1) (**Covering f -local stable sets**) for any $z = (x, y) \in U(z_n, r^{(n)})$, if $\hat{\mathcal{U}}^{n,s}(z)$ denotes the atom of $\hat{\mathcal{U}}^{n,s}$ containing z , then

$$\text{pr}_1\left(\hat{\mathcal{U}}^{n,s}(z)\right) = W_{2r^{(n)}}^s(x) \cap B_M(x_n, r^{(n)}).$$

- (2) (**Local invariance**) for every $n \in \{0, \dots, N_0 - 1\}$ and any $z \in U(z_n, \frac{r^{(n)}}{C})$,

$$F(z) \in U(z_{n+1}, r^{(n+1)}),$$

and

$$F\left(\hat{\mathcal{U}}^{n,s}(z)\right) \cap U(z_{n+1}, r^{(n+1)}) \subset \hat{\mathcal{U}}^{n+1,s}(F(z)).$$

- (3) (**Uniform contraction**) for every $k \in \{1, \dots, N_0\}$ and any pair of points $z', z'' \in \hat{\mathcal{U}}^{0,s}(z_0) \cap B_M\left(x_0, \frac{r^{(0)}}{CC}\right) \times B_N\left(y_0, \frac{r^{(0)}}{CC}\right)$,

$$(29) \quad d(F^k(z'), F^k(z'')) \leq \tilde{C} e^{-k\kappa} d(z', z'').$$

The partitions $\hat{\mathcal{U}}^{n,s}$ given by Proposition 4.2 will be called a *fake locally stable foliations* and their elements *fake locally stable sets*. Similarly, we can define the *fake locally unstable foliations* exhibiting analogous properties.

The rest of this section is devoted to the proof of Proposition 4.2. First we need some lemmas:

Lemma 4.3. *Let $\mathbb{R}^q = \mathbb{R}^u \oplus \mathbb{R}^c \oplus \mathbb{R}^s$ be an orthogonal splitting and $L_i: \mathbb{R}^q \hookrightarrow$, with $i \in \{1, \dots, n\}$, be linear maps that preserve the splitting. Suppose there exists $\lambda \in (0, 1)$ such that it holds*

$$\begin{aligned} \max \left\{ \|L_i|_{\mathbb{R}^s}\|, \|(L_i|_{\mathbb{R}^u})^{-1}\| \right\} &< \lambda; \\ \max \left\{ \|L_i|_{\mathbb{R}^c}\|, \|(L_i|_{\mathbb{R}^c})^{-1}\| \right\} &< \lambda^{-\frac{1}{2}}, \end{aligned}$$

for each $i \in \{1, \dots, n\}$.

Then there exists a number $\alpha_2 = \alpha_2(\lambda) > 0$ such that if $f_i: \mathbb{R}^q \hookrightarrow$ are C^1 -diffeomorphisms satisfying $d_{C^1}(f_i, L_i) < \alpha_2$ for every $i \in \{1, \dots, n\}$, then there exists a family of foliations \mathcal{W}_i^s of \mathbb{R}^q , called *stable foliations*, satisfying the following properties:

- (1) $f_i(\mathcal{W}_i^s) = \mathcal{W}_{i+1}^s$, for each $i \in \{1, \dots, n-1\}$;
(2) for any $x \in \mathbb{R}^q$, any $i \in \{1, \dots, n-1\}$ and every $y \in \mathcal{W}_i^s(x)$,

$$\|f_i^k(x) - f_i^k(y)\| < \lambda^k \|x - y\|, \quad \forall k \in \{1, \dots, n-i\}.$$

An analogous result holds for unstable foliations.

Proof. This is essentially Hadamard-Perron's theorem. See [KH96, Theorem 6.2.8] for details. \square

Lemma 4.4. *Let δ, η and ℓ be positive numbers and suppose that $e^\eta < \frac{4}{3}$. Let $f, g: \mathbb{R}^q \hookrightarrow$ be C^1 -diffeomorphisms such that $d_{C^1}(f, g) < \delta$. Let $A, B \in \text{GL}(q, \mathbb{R})$ be two linear transformations such that either $\|A\| < \ell e^\eta$ and $\|B\| < \ell$, or $\|A\| < \ell$ and $\|B\| < \ell e^\eta$. Then,*

$$(30) \quad d_{C^1}(AfB, AgB) < 4\ell^2\delta.$$

Proof. The estimate easily follows from the following computation:

$$\begin{aligned} & d_{C^1}(AfB(\cdot), AgB(\cdot)) \\ &= d_{C^0}(AfB(\cdot), AgB(\cdot)) + d_{C^0}(A \circ Df(B(\cdot)) \circ B(\cdot), A \circ Dg(B(\cdot)) \circ B(\cdot)) \\ &\leq \|A\| d_{C^0}(f, g) + \|A\| d_{C^0}(Df, Dg) \|B\| + \|A\| d_{C^0}(f, g) \|B\| \\ &\leq 3\ell^2 e^\eta \delta < 4\ell^2 \delta. \end{aligned}$$

□

For any small number $r > 0$, we can choose a C^1 map $\rho_r : \mathbb{R}^q \rightarrow [0, 1]$ such that $\rho_r = 1$ on $B(0, r)$ and $\rho_r = 0$ on $\mathbb{R}^q \setminus B(0, 2r)$, and $\|\rho_r\|_{C^0} \leq 1$, $\|D\rho_r\| = O(\frac{1}{r})$. Then, consider any $C^{1+\beta}$ function $f : N \hookrightarrow$, where N is a closed smooth Riemannian manifold. Define a diffeomorphism $f^r : TN \rightarrow TN$ as follows,

$$(31) \quad f^r(x, v) = \rho_r(v) \times \exp_{f(x)}^{-1} \circ f \circ \exp_x(v) + (1 - \rho_r(v)) \times Df_x(v).$$

Note that

$$(32) \quad f^r(x, v) := \begin{cases} \exp_{f(x)}^{-1} \circ f \circ \exp_x(v), & \text{if } \|v\| \leq r, \\ Df_x(v), & \text{if } \|v\| \geq 2r. \end{cases}$$

Then we have the following C^1 estimate for this diffeomorphism.

Lemma 4.5.

$$(33) \quad d_{C^1}(f^r(x, \cdot), Df_x(\cdot)) = O(r^\beta).$$

Proof. By definition,

$$(34) \quad f^r(x, v) - Df_x(v) = \rho_r(v) \times (\exp_{f(x)}^{-1} \circ f \circ \exp_x(v) - Df_x(v)).$$

Then, for r small,

$$\begin{aligned} (35) \quad & \|f^r(x, \cdot) - Df_x(\cdot)\|_{C^1} \\ &\leq \|\rho_r(\cdot) \times (\exp_{f(x)}^{-1} \circ f \circ \exp_x(\cdot) - Df_x(\cdot))\|_{C^0} \\ &\quad + \|D\rho_r(\cdot) \times (\exp_{f(x)}^{-1} \circ f \circ \exp_x(\cdot) - Df_x(\cdot))\|_{C^0} \\ &\quad + \|\rho_r(\cdot) \times D(\exp_{f(x)}^{-1} \circ f \circ \exp_x(\cdot) - Df_x(\cdot))\|_{C^0} \\ &\leq (2r)C(2r)^\beta + \frac{C}{r}(2r)C'(2r)^\beta + C'(2r)^\beta \\ &\leq C(r^{1+\beta} + r^\beta + r^\beta) < Cr^\beta. \end{aligned}$$

□

Proof of Proposition 4.2. Let $\lambda'_1 > \dots > \lambda'_k$ be the distinct fibered Lyapunov exponents of the linear cocycle B defined in (22), with multiplicities n_1, \dots, n_k , respectively.

Fix a number

$$(36) \quad \kappa < \frac{1}{2} \min_i \{\lambda'_i - \lambda'_{i+1}, \tau\},$$

where τ is the hyperbolicity rate of $f : M \rightarrow M$. With respect to the number $e^{-\kappa} \in (0, 1)$, Lemma 4.3 gives the constant α_2 . Then we choose the number η such that

$$(37) \quad \eta < \min \left\{ \beta^2 \kappa, \frac{1}{2} \min_i \{ \lambda'_i - \lambda'_{i+1} \} - \kappa, \frac{\alpha_2}{2} \right\}.$$

Invoking Theorem 4.1, we know there is a measurable map $C_\eta : \Lambda_\eta \rightarrow \text{GL}(q, \mathbb{R})$, a function $K : \Lambda_\eta \rightarrow (0, +\infty)$, with $K(z) > \max\{\|C_\eta(z)\|, \|C_\eta^{-1}(z)\|\}$, such that

$$(38) \quad e^{-\eta} < \frac{K(z)}{K(F(z))} < e^\eta.$$

For any small $r > 0$, $z = (x, y) \in \Lambda_\eta$, we can define F_z^r as follows

$$(39) \quad F_z^r(x', v) = \begin{cases} \exp_{\text{pr}_2 \circ F(z)}^{-1} \circ F(x', \exp_y(v)), & \text{if } \|v\| < r, \\ D(A(x'))(y)(v), & \text{if } \|v\| > 2r. \end{cases}$$

For sufficiently $r > 0$ sufficiently small $B(x, r) \subset M$, F_z^r is well-defined on $B(x, r) \times \mathbb{R}^q$. Moreover, by Lemma 4.5, for some constant K ,

$$(40) \quad \|F_z^r(x, \cdot) - B(z)(\cdot)\|_{C^1} \leq Kr^\beta.$$

Note that for some ℓ sufficiently large, the uniformity block $\Lambda_{\eta, \ell} \subset \Lambda_\eta$ has positive μ -measure. For any orbit segment $\{z_0, \dots, z_{N_0}\}$, with $z_0, z_{N_0} \in \Lambda_{\eta, \ell}$, Theorem 4.1 item (1) gives N_0 linear maps

$$(41) \quad B_\eta(z_0), B_\eta(z_1), \dots, B_\eta(z_{N_0-1}),$$

all with Lyapunov block forms.

Now we take

$$(42) \quad r^{(0)} < \min \left\{ \frac{1}{\ell^2}, \left(\frac{\alpha_2}{8K\ell^2} \right)^{1/\beta} \right\}.$$

Then by (40),

$$(43) \quad \|F_{z_0}^{r^{(0)}}(x_0, \cdot) - B(z_0)(\cdot)\|_{C^1} < \alpha_2/8\ell^2,$$

and by our choice of the uniformity block,

$$(44) \quad \max(\|C_\eta(z_0)\|, \|C_\eta(z_0)^{-1}\|) \leq K(z_0) \leq \ell,$$

$$(45) \quad \max(\|C_\eta(z_1)\|, \|C_\eta(z_1)^{-1}\|) \leq K(z_1) \leq \ell e^\eta.$$

By Lemma 4.4, we get

$$(46) \quad \|C_\eta(z_1) \circ F_{z_0}^{r^{(0)}}(x_0, \cdot) \circ C_\eta^{-1}(z_0) - B_\eta(z_0)\|_{C^1} < \alpha_2/2.$$

So, it follows from Theorem 4.1 item (2) that, for any $x' \in B(x_0, r^{(0)})$, it holds

$$(47) \quad \|C_\eta(z_1) \circ F_{z_0}^{r^{(0)}}(x', \cdot) \circ C_\eta^{-1}(z_0) - B_\eta(z_0)\|_{C^1} < \alpha_2.$$

Now, let us consider the map $F_{z_n}^{r^{(n)}}(x_n, \cdot)$, for $n \in \{1, \dots, N_0 - 1\}$.

By definition of $r^{(n)}$ in (28), and noting (25), it follows that for any $n = 0, \dots, N_0$, $U(z_n, r^{(n)})$ is contained in the regular neighborhood $B_M(x_n, r_1(z)) \times \Phi_{z_n}^{-1}(B(0, r_2(z)))$. Moreover,

$$(48) \quad r^{(n)} < r^{(0)} e^{-\frac{\eta}{\beta} \min\{n, N_0 - n\}} < \left(\frac{\alpha_2}{8K\ell^2} \right)^{\frac{1}{\beta}} e^{-\frac{\eta}{\beta} \min\{n, N_0 - n\}} = \left(\frac{\alpha_2}{8K\ell^2 e^{\eta \min\{n, N_0 - n\}}} \right)^{\frac{1}{\beta}}.$$

By Lemma 4.5,

$$(49) \quad \|F_{z_{n-1}}^{r^{(n-1)}}(x_{n-1}, \cdot) - B(z_{n-1})(\cdot)\|_{C^1} < \frac{\alpha_2}{8\ell^2} e^{-\eta \min\{n, N_0 - n\}}.$$

Noting that $K(z_{n-1}) \leq \ell e^{\eta \min\{n, N_0-n\}}$ and $K(z_n) \leq \ell e^{\eta \min\{n+1, N_0-n-1\}}$, and applying Lemma 4.4, we obtain

$$(50) \quad \|C_\eta(z_n) \circ F_{z_{n-1}}^{r^{(n-1)}}(x_{n-1}, \cdot) \circ C_\eta^{-1}(z_{n-1}) - B_\eta(z_{n-1})\|_{C^1} < \frac{\alpha_2}{2},$$

Then it follows from Theorem 4.1 that for any $x' \in B(x_{n-1}, r^{(n-1)})$,

$$(51) \quad \|C_\eta(z_n) \circ F_{z_{n-1}}^{r^{(n-1)}}(x', \cdot) \circ C_\eta^{-1}(z_{n-1}) - B_\eta(z_{n-1})\|_{C^1} < \alpha_2.$$

Now, by Lemma 4.3 and the above estimates, the mappings $\{C_\eta(z_n) \circ F_{z_{n-1}}^{r^{(n-1)}}(x', \cdot) \circ C_\eta^{-1}(z_{n-1})\}_{n=1}^{N_0}$ admit stable foliations of \mathbb{R}^d . We write these stable foliations as $\mathcal{W}^{n-1, s, x'}$.

The choice of η in (37) gives $2\kappa + 2\eta < \min_i \{\lambda'_i - \lambda'_{i+1}\}$. For any $w \in \mathcal{W}^{n, s, x'_n}(v)$, where $x'_n \in B(x_n, r^{(n)})$, the forward iterates of x'_n will be denoted by x'_{n+1}, \dots, x'_j . For $N_0 \geq j \geq n$,

$$(52) \quad \begin{aligned} d_E \Big(C_\eta(z_{j+1}) \circ F_{z_j}^{r^{(j)}}(x'_j, \cdot) \circ \dots \circ F_{z_n}^{r^{(n)}}(x'_n, \cdot) \circ C_\eta(z_n)^{-1}(w), \\ C_\eta(z_{j+1}) \circ F_{z_j}^{r^{(j)}}(x'_j, \cdot) \circ \dots \circ F_{z_n}^{r^{(n)}}(x'_n, \cdot) \circ C_\eta(z_n)^{-1}(v) \Big) \\ < e^{-2(j-n)\kappa} d_E(w, v), \end{aligned}$$

where we use d_E to denote the standard Euclidean distance.

We can now define the stable partition $\hat{\mathcal{U}}^{n, s}$ of $U(z_n, r^{(n)})$. It is done in two steps. Firstly, we consider the stable foliation W^s of the ball $B(x_n, r^{(n)}) \subset M$. Then, on each fiber over $x \in B(x_n, r^{(n)})$, the set $B(y_n, r^{(n)})$ is foliated by the the projection of the foliation $\mathcal{W}^{n, s, x}$ constructed above, i.e., the foliation $\exp_{y_n} \circ C_\eta(z_n)^{-1}(\mathcal{W}^{n, s, x})$ restricted to $B(y_n, r^{(n)})$. Note item (1) of the proposition follows directly from this definition.

For all $n = 0, \dots, N_0$, $r^{(n+1)} \geq r^{(n)} e^{-\frac{\eta}{\beta^2}}$, it is possible to choose a large constant C , such that

$$(53) \quad F\left(U\left(z_n, \frac{r^{(n)}}{C}\right)\right) \subset U(z_{n+1}, r^{(n+1)}).$$

This shows Item (2).

Now we prove item (3). Define $\tilde{C} = \max\{\frac{\ell^2}{2}, K_0\}$, where K_0 is the constant in Definition 2.1. Now we take points $z', z'' \in \hat{\mathcal{U}}^{0, s}(z_0) \cap B_M\left(x_0, \frac{r^{(0)}}{C\tilde{C}}\right) \times B_N\left(y_0, \frac{r^{(0)}}{C\tilde{C}}\right)$, and we will write $z'_k = (x'_k, y'_k) = F^k(z')$, $z''_k = (x''_k, y''_k) = F^k(z'')$ for all k . Inductively, both points z'_k and z''_k are contained in $\hat{\mathcal{U}}^{k, s}(z_k) \cap B_M\left(x_k, \frac{r^{(k)}}{C}\right) \times B_N\left(y_k, \frac{r^{(k)}}{C}\right)$. Then item (2) ensures the estimate (52) can be applied at each step.

Thus, by hyperbolicity of the base dynamics,

$$(54) \quad d_M(x'_k, x''_k) \leq K_0 e^{-\tau k} d_M(x'_0, x''_0) \leq \frac{r^{(k)}}{C}.$$

The second coordinate has a similar estimate. Indeed, it follows from (52) that, for any $k = 1, \dots, N_0 - 1$,

$$(55) \quad d(y'_k, y''_k) \leq d(y'_k, y_k) + d(y''_k, y_k).$$

We estimate

$$\begin{aligned}
d(y'_k, y_k) &\leq \ell e^{k\eta} d_E \{C_\eta(z_k) \circ F_{z_{k-1}}^{r^{(k-1)}}(x'_{k-1}, \cdot) \circ \cdots \circ F_{z_0}^{r^{(0)}}(x'_0, \exp_{y_0}^{-1}(y'_0))\}, \\
&\quad C_\eta(z_k) \circ F_{z_{k-1}}^{r^{(k-1)}}(x_{k-1}, \cdot) \circ \cdots \circ F_{z_0}^{r^{(0)}}(x_0, 0)\} \\
&\leq \ell e^{-k\kappa} d(C_\eta(z_0) \circ \exp_{y_0}^{-1}(y'_0), 0) \\
&\leq \ell^2 e^{-k\kappa} d(y'_0, y_0) \leq \frac{r^{(k)}}{2C}.
\end{aligned}$$

The term $d(y'_k, y_k)$ satisfies a similar estimate. So we have $d(y'_k, y''_k) \leq \frac{r^{(k)}}{C}$. Combining the first and the second coordinate estimates, we have finished the proof of item (3) of the proposition. \square

5. FIBER CLOSING LEMMA AND PROOF OF MAIN RESULTS

In this section we prove Theorem 1.1. We start with a technical result which is some kind of “Closing Lemma along the fibers”.

Let $F: M \times N \hookrightarrow$ be the skew-product induced by the hyperbolic homeomorphism $f: M \hookrightarrow$ and the C^β -cocycle $A: M \rightarrow \text{Diff}^{1+\beta}(N)$. Let μ be an ergodic F -invariant probability measure. Let η, ℓ be positive numbers with ℓ large enough in order to guarantee that the Pesin uniformity block $\Lambda_{\eta, \ell}$ of F has positive μ -measure. Let $r^{(0)}, C, \tilde{C}, \kappa$ be the constants given by Proposition 4.2. In what follows, when there is no confusion, we denote the iterates of a point $z_0 = (x_0, y_0) \in M \times N$ by $z_i = (x_i, y_i) = F^i(z_0)$, for every $i \in \mathbb{Z}$. Since $\mu(\Lambda_{\eta, \ell}) > 0$, notice that by Poincaré recurrence theorem, there exists $z_0 \in \Lambda_{\eta, \ell}$ such that z_n belongs to $\Lambda_{\eta, \ell}$ too, for some $n \geq 1$.

Lemma 5.1. *There exist constants $\varepsilon_0, K > 0$ such that for any point $z_0 \in \Lambda_{\eta, \ell}$ such that there is $n \geq 1$ with $z_n \in \Lambda_{\eta, \ell}$ and $d(z_0, z_n) < \varepsilon_0$, there is some F -orbit segment $\{z''_0, z''_1, \dots, z''_n\}$, with $\text{pr}_1(z''_0) = \text{pr}_1(z''_n)$ and satisfying*

$$(56) \quad d(z''_i, z_i) \leq K d(z_0, z_n) e^{-\kappa \min\{i, n-i\}}.$$

Proof. Let us define $K = \max\{\ell, 2C\tilde{C}\}$ and $\varepsilon_0 = r^{(0)}/K$.

First note that for any orbit segment satisfying $d(z_0, z_n) < \varepsilon_0$ and $z_0, z_n \in \Lambda_{\eta, \ell}$, the point z_n lies in the regular neighborhood of z_0 .

In particular, $d(x_0, x_n) < \varepsilon_0$ and hence we can apply Anosov Closing Lemma to conclude there is a unique periodic point p with period n which exponentially shadows the orbit segment $\{x_0, \dots, x_n\}$. Consider the point $x'_0 = W^s(x_0) \cap W^u(p)$. Then, for each $i = 0, \dots, n$ it holds

$$(57) \quad d(x_i, x'_i) < \frac{K}{2} e^{-\tau \min\{i, n-i\}} d(x_0, x_n).$$

Since $x'_0 \in W_{\frac{r^{(0)}}{K}}^s(x_0)$, by Proposition 4.2, x'_0 has a pre-image by pr_1 which is contained in the local fake stable set of z_0 . Denote this point by $z'_0 \in \hat{\mathcal{U}}^{0,s}(z_0)$. Moreover, by our choice of K it holds

$$(58) \quad d(z_i, z'_i) < \frac{K}{2} d(z_0, z'_0) e^{-\kappa \min\{i, n-i\}}, \quad \forall i \in \{1, \dots, n\}$$

In particular, $d(z'_n, z_n) < r^{(n)}$. Notice that $r^{(n)} = r^{(0)}$, too.

Now let us consider the point $x'_n = \text{pr}_1(z'_n) \in M$. Since $d(p, x'_n) \leq d(p, x_n) + d(x_n, x'_n)$, the periodic point $p \in W_{2\varepsilon_0}^u(x'_n)$. Applying Proposition 4.2 in the backward direction, we conclude the point p has a pre-image under pr_1 , denoted as

$z_n'' \in \widehat{\mathcal{U}}^{n,u}(z_n')$, such that if we write $z_i'' := F^{-(n-i)}(z_n'')$ for all $i = 0, \dots, n-1$, then it holds

$$(59) \quad d(z_i'', z_i') < \frac{K}{2} d(z_n'', z_n') e^{-\kappa \min\{i, n-i\}}.$$

Combing both (58) and (59), for all $i = 0, \dots, n$,

$$(60) \quad d(z_i'', z_i) \leq d(z_i'', z_i') + d(z_i', z_i) \leq K d(z_n, z_0) e^{-\kappa \min\{i, n-i\}},$$

where the last inequality is consequence of the fact that $d(z_0, z_n) \sim d(z_0, z_0')$ and $d(z_0, z_n) \sim d(z_n'', z_n')$, maybe increasing K if it is necessary. Finally it clearly holds $\text{pr}_1(z_n'') = \text{pr}_1(z_0'') = p$. \square

Proof of Theorem 1.1. According to Theorem 3.1 of [KP16], under the periodic orbit condition (3), it suffices to show that the fiber-wise Lyapunov exponents all vanish, with respect to any ergodic F -invariant probability measure μ .

Reasoning by contradiction, suppose that there exists an ergodic F -invariant measure μ such that their fibered Lyapunov exponents, listed by $\lambda_1 \geq \dots \geq \lambda_q$, are not all equal to zero. If the measure is supported on finitely many fibers, then we immediately get a contradiction, because the periodic orbit condition clearly implies all Lyapunov exponents vanish. If this is not the case, we consider a positive number δ which is smaller than the absolute value of one non-vanishing Lyapunov exponent.

We claim that there exists an orbit segment z_0'', \dots, z_n'' with $\text{pr}_1(z_0'') = \text{pr}_1(z_n'')$, such that, if we denote the singular values of $DA^{(n)}(x_0'')(y_0'')$ by $e^{n\sigma_1} \geq \dots \geq e^{n\sigma_d}$, then they satisfy $|\sigma_j - \lambda_j| \leq \delta$ for all $1 \leq j \leq d$. In such a case, this linear map has at least one non-vanishing singular value, contradicting the periodic orbit condition (3).

So, it just remains to prove our claim. Let κ be a positive real number satisfying estimate (36), and let $\alpha_1 = \alpha_1(\delta/2)$ be the positive constant given by Lemma 3.3. Then, we apply Lemma 4.3 getting a new constant $\alpha_2 := \alpha(e^{-\kappa})$; and we take $\alpha := \min\{\alpha_1, \alpha_2\}$. Thus, let η be a positive number satisfying (37). By Theorem 4.1, we can choose a uniformity block $\Lambda_{\eta, \ell}$ with positive μ -measure. Finally, let ε_0 and K be the positive constants given by Lemma 5.1.

Then, we choose a point z in $\Lambda_{\eta, \ell}$ which is a μ -density point of $\Lambda_{\eta, \ell}$ and consequently, the ball $B(z, \varepsilon_0/2)$ intersects $\Lambda_{\eta, \ell}$ with positive μ -measure. By Poincaré's recurrence theorem, we can assume there are infinitely many natural numbers n such that the F -orbit of z satisfies $z_0, z_n \in B(z, \varepsilon_0/2) \cap \Lambda_{\eta, \ell}$.

Applying Lemma 5.1, there is an F -orbit segment $z_0'', z_1'', \dots, z_n''$, with $\text{pr}_1(z_0'') = \text{pr}_1(z_n'')$, such that,

$$(61) \quad d(z_i, z_i'') < K d(z_0, z_n) e^{-\kappa \min\{i, n-i\}} \text{ for all } i = 0, \dots, n.$$

Thus, by formula (50) we know that

$$(62) \quad \|C_\eta(z_i) \circ F_{z_{i-1}}^{r(i-1)}(z_{i-1}, \cdot) \circ C_\eta^{-1}(z_{i-1})^{-1} - B_\eta(z_{i-1})\|_{C^1} < \alpha.$$

So, if $\tilde{\sigma}_1^{(n)} \geq \tilde{\sigma}_2^{(n)} \geq \dots \geq \tilde{\sigma}_q^{(n)}$ denotes the singular values of the linear map $C_\eta(z_n) \circ DA^n(x_0'')(y_0'') \circ C_\eta(z_0)^{-1}$, then invoking Lemma 3.3 for the constant $\delta/2$, we can guarantee that

$$\left| \frac{1}{n} \log \tilde{\sigma}_j^{(n)} - \lambda_j \right| < \frac{\delta}{2},$$

for each $i \in \{1, \dots, q\}$.

Finally, note that the changes of Lyapunov coordinates at z_0 and z_n , namely $C_\eta(z_0)$ and $C_\eta(z_n)$ and their inverses, are bounded by ℓ . So we can apply Lemma 3.2

to conclude that, if $\sigma_1^{(n)} \geq \sigma_2^{(n)} \geq \dots \geq \sigma_q^{(n)}$ are the singular values of the linear map $DA^n(x_0'')(y_0'')$, then it holds

$$\left| \frac{1}{n} \log \tilde{\sigma}_j^{(n)} - \lambda_j \right| < \delta,$$

for every i , completing the proof of the theorem. \square

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